$$A_i(2m) = A_i,\ C_i(2m) = C_i$$

$$B_i(2m) = B_i + \begin{bmatrix} -\Delta^2\gamma\lambda^4 f_i & -\Delta^2\gamma\lambda^4 g_i \\ \Delta^2\gamma g_i & 0 \end{bmatrix}$$

$$D_i(2m) = \begin{bmatrix} f_i g_i \Delta^2\gamma\lambda^4 \\ -\frac{1}{2} g_i g_i \Delta^2\gamma \end{bmatrix}$$

$$E_i(2m) = \begin{bmatrix} \gamma\lambda^4 f_i(2m) g_i(2m)\Delta^2 \\ -\frac{1}{2}\gamma g_i(2m) g_i(2m)\Delta^2 \end{bmatrix}$$

where now

$$T_j = \begin{bmatrix} g_j \\ f_j \end{bmatrix} = T_j(1) + T_j(3) + \ldots + T_j(2m-1) = \begin{bmatrix} g_j(1) + g_j(3) + \ldots + g_j(2m-1) \\ f_j(1) + f_j(3) + \ldots + f_j(2m-1) \end{bmatrix} \quad (15)$$

The iteration is repeated until the remainder terms $T_j(2m)$ become small compared with the accumulated solution $T_j(1) + T_j(3) + \ldots + T_j(2m-1)$. The main advantage of the modified method is that after the first iteration some features of the nonlinearity are included in the linear equations, whereas in Archer's approach the particular nature of the nonlinear terms is relatively unimportant at a given solution step.

Reference 3 contains a comprehensive summary of the stresses and displacements in a spinning shallow spherical shell obtained by using Reissner's linear[8] and nonlinear[6] theories for a significant range of shell geometry and inertia loading parameters. Figures 2 and 3 show the variation of two of the stress components, the nondimensional radial direct stress and radial bending stress resultants, respectively, for a shell with fairly substantial curvature.

Note in Fig. 2 that for low γ or low inertia loading, the nonlinear and linear solutions nearly coincide whereas for large γ the solution is nearly that of a flat disk. For small γ, a membrane stress distribution is approached. As the inertia load increases, the shell progressively flattens out and the stress distribution alters radically, until, finally, the stress distribution is essentially the same as if the shell had been flat to start with. That is, the stress required to overcome the initial curvature is small compared to the added stress built up after the disk is almost flat.

As shown in Fig. 3, nonlinear effects appear in the bending stress resultant at even small inertia loadings. The effect of increasing γ is to decrease the bending stress. For high γ the residual bending stress becomes small as the membrane effects predominate.

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Unsteady Three-Dimensional Stagnation-Point Flow

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Nomenclature

f, F, g, G	=	nondimensional velocity functions
n	=	$(\beta/\alpha)^{1/2}$
p	=	pressure
P	=	nondimensional pressure function
t	=	time
u, v, w	=	x, y, z velocity components, respectively
x	=	coordinate tangent to surface
y	=	coordinate normal to surface
z	=	coordinate normal to x, y plane
α, β, γ	=	constants in velocity components
η, ξ	=	similarity variables in y and t
λ	=	acceleration parameter
ν	=	kinematic viscosity
ρ	=	density

Introduction

In a recent Note, Williams[1] found exact solutions to the Navier-Stokes equations for a special class of unsteady incompressible flows in the vicinity of either a two-dimensional or an axisymmetric stagnation point. These solutions are obtained by solving a single nonlinear, ordinary differential equation. In the present Note, we show, by extending the work of Howarth[2] on the boundary layer in three-dimensional flow, that exact solutions to the Navier-Stokes equations are possible for a similar class of unsteady incompressible flows in the vicinity of a stagnation point on a general (three-dimensional) surface. In this general case, the problem reduces to two simultaneous nonlinear, ordinary differential equations containing a parameter; for particular values of this parameter, the two-dimensional and axisymmetric cases are regained.

Analysis

Consider an infinite plate in the x-z plane. There is a stagnation point at $x = y = z = 0$. The class of three-dimensional (steady) flows considered by Howarth is characterized by $u \sim x$, $w \sim z$, and $v \sim y$ as $y \to \infty$, and is determined by solving the Navier-Stokes equations exactly. The two-dimensional unsteady flow considered by Williams is characterized by $u \sim x/(1 + \lambda t)$ and $v \sim y/(1 + \lambda t)$ as $y \to \infty$. His unsteady axisymmetric case exhibits a similar time dependence. Here, guided by the aforementioned considerations, we incorporate the unsteady similarity variable of Williams' work with the three-dimensional form used by Howarth. Thus, for a three-dimensional, unsteady stagnation-point flow, the velocity components are assumed to be

$$u = \alpha x F'(\xi)/(1 + \lambda t),\ w = \beta z G'(\xi)/(1 + \lambda t) \quad (1)$$

and

$$v = -[\nu/\gamma(1 + \lambda t)]^{1/2}\{\alpha F(\xi) + \beta G(\xi)\} \quad (2)$$

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in which

$$\gamma^{1/2} = \alpha^{1/2} + \beta^{1/2}$$

and

$$\xi = [\gamma/\nu(1 + \gamma t)]^{1/2} y$$

In Eqs. (1) (and hereafter), prime signifies ordinary differentiation with respect to the appropriate independent variable. In keeping with the usual analysis of stagnation-point flows, the pressure field is assumed to vary from its inviscid form only in the direction normal to the surface so that, introducing a function $P(\xi)$, let

$$p - p_0 = -[\rho/2(1 + \lambda t)^2]\{x^2(\alpha^2 - \alpha\lambda) + z^2(\beta^2 - \beta\lambda)\} - [\rho\nu\gamma/(1 + \lambda t)]P(\xi) \quad (3)$$

where p_0 is the stagnation pressure.

In order to determine the unknown functions F, G, and P, the three Navier-Stokes equations of motion are available. (The equation of continuity for incompressible flow is identically satisfied by the assumed velocity field.) Substitution of the assumed velocity components and pressure field into the x and z components of the Navier-Stokes equations yields, respectively,

$$-\lambda(1 + n)^2/\gamma\{-1 + F' + (\xi/2)F''\} + F'^2 - FF'' - n^2GF'' = 1 + (1 + n)^2F''' \quad (4)$$

and

$$-\lambda(1 + n^{-1})^2/\gamma\{-1 + G' + (\xi/2)G''\} + G'^2 - GG'' - n^{-2}FG'' = 1 + (1 + n^{-1})^2G''' \quad (5)$$

where $n^2 = \beta/\alpha$. Similarly, the y component gives, after a simple rearrangement,

$$P' = \lambda(1 + n)^{-2}/2\gamma\{\xi F + n^2\xi G\}' + (1 + n)^{-4}/2\{(F + n^2G)^2\}' + (1 + n)^{-2}\{F'' + n^2G''\} \quad (6)$$

The boundary conditions require a stagnation point (for all n^2) at $x = y = z = 0$ and that as $y \to \infty$ (or $\xi \to \infty$), u and w approach the inviscid velocity components, i.e., $u \to \alpha x/(1 + \lambda t)$ and $w \to \beta z/(1 + \lambda t)$. Hence, from Eqs. (1–3),

$$F(0) = F'(0) = 0, F'(\xi \to \infty) = 1 \quad (7)$$

$$G(0) = G'(0) = 0, G'(\xi \to \infty) = 1 \quad (8)$$

and

$$P(0) = 0 \quad (9)$$

It is easy to show that (4) and (5) include the results of both Howarth and Williams as special cases. With new variables defined by $\eta = \xi/(1 + n)$, $f(\eta) = F(\xi)/(1 + n)$, and $g(\eta) = G(\xi)/(1 + n)$, (4) and (5) are

$$-(\lambda/\alpha)\{-1 + f' + (\eta/2)f''\} + f'^2 - ff'' - n^2gf'' = 1 + f''' \quad (10)$$

and

$$-(\lambda/\alpha)\{-1 + g' + (\eta/2)g''\} + n^2g'^2 - n^2gg'' - fg'' = n^2 + g''' \quad (11)$$

For $\lambda = 0$, these equations reduce to those of Howarth (his c is our n^2). For $n = 0$ ($\beta = 0$), only (10) remains since this is the two-dimensional case ($w \equiv 0$); its form is precisely that of Williams' two-dimensional case. For $n = 1$, (10) and (11) are identical so that $f = g$ and either equation is Williams' axisymmetric case.

To determine the pressure field in terms of F and G, we integrate (6) directly and make use of (9) to obtain

$$P(\xi) = \lambda(1 + n)^{-2}/2\gamma\{\xi F + n^2\xi G\} + (1 + n)^{-4}/2\{F + n^2G\}^2 + (1 + n)^{-2}\{F' + n^2G'\}$$

A (numerical) solution of (4) and (5) is simplified somewhat since n need be considered only in the range $0 < n < 1$. This follows by noting that replacement of n by $1/n$ does not change the system of equations or γ. Preliminary numerical work indicates that a complete study of the solutions $F(\xi; n, \lambda/\gamma)$ and $G(\xi; n, \lambda/\gamma)$ over the ranges of the parameters would be lengthy. However, since solutions are known for particular values of the parameters, the method of parametric differentiation proposed by Rubbert and Landahl[3] appears to be pertinent. It is hoped that results obtained with this method can be reported at a later date.

The aforementioned flow can be used, for example, to study a particular unsteady, viscous flow near the stagnation point of an ellipsoid with one of its axes along the direction of flow. The required values of α and β are determined by expanding the inviscid fluid velocity components in the neighborhood of the stagnation point. This method of determining α and β indicates the solution is good in the neighborhood of the stagnation point on any body in the particular unsteady flow considered, provided that the surface near that point is smooth.

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A Second-Order Correction to the Glauert Wall Jet Solution

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Recently, much interest has been shown in the problem of a plane, laminar, incompressible jet flowing along a curved surface with no external stream present. Some applications are in boundary-layer control and circulation control of airfoils. A similarity solution for the straight wall case was found by Glauert.[1] Lindow and Greber[2] have found a similarity solution for a particular curved wall geometry but have not included the additional second-order effect due to displacement (See Ref. 3, p. 276). Rubin and Falco[4] calculated the displacement effect for a plane freejet but as yet no second-order solution including displacement has been published for the flow situation with a wall present. In this paper, this effect will be found for the Glauert wall jet.

Let L be an arbitrary reference length and U an arbitrary reference speed. The Reynolds number R is given by $R = UL/\nu$, where ν is the kinematic viscosity of the fluid. R is chosen sufficiently large so that second-order boundary-layer theory as developed by Van Dyke[3] is applicable.

First-Order Jet Solution

x and y denote distances along and normal to the wall, x being measured from the jet slot and have been nondimensionalized with L. The Navier-Stokes equations for the

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